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On gradient dynamical system on semi-Riemannian manifolds

JENS CHR. LARSEN
Mathematical Institute
The Technical University of Denmark
Building 303, DK-2800 Lyngby, Denmark

Abstract. *We consider gradient dynamical systems on a semi-Riemannian manifold of arbitrary index. The main point of the paper is the introduction of the concepts causality subsets, causality function and sector stability. As a main application we provide conditions assuring, that the nonwandering points are precisely the singular points of the gradient field. Furthermore we show, that every nonconstant recurrent orbit for the gradient field must intersect one of the causality subsets and that the stable and unstable manifolds belonging to a hyperbolic singular point for the gradient field are orthogonal.*

0. INTRODUCTION

In [6] S. Smale poses the problem: «What can one say about the dynamical systems, which are gradient systems of a function with respect to a non-degenerate indefinite metric, say on a compact manifold?»

In this paper we pose the question in the broader context of arbitrary semi-Riemannian manifolds and develop a theory which among others enable us to address the question of Smale.

The question of Smale is not only of mathematical interest but has direct physical interest, since this problem as shown in [6] is strongly related to the theory of electrical networks. The question also naturally arises in the theory of relativity.

Key-Words: Gradient dynamical system, semi-Riemmanian manifold, recurrent orbit, nonwandering point, Killing vector field.
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The main part of the paper will be devoted to the development of the concepts causality function, causality subsets and sector stability. These concepts are important for a description of the dynamics. Under suitable conditions, involving the causality subsets and the causality function, semi-Riemannian gradient dynamical systems are simple in the sense, that the recurrent points resp. nonwandering points are precisely the singular points of the gradient. This is the content of the main theorems 1.2. and 1.3. In Proposition 1.6. we show how the causality subsets relate to a recurrent orbit for the gradient field.

In section 2 we define sector stability. As an example of the importance of sector stability consider a vector field, defined on all of $\mathbf{R}^3$, in variables x_A, x_B and x_C, the concentrations of chemical reactants A, B and C respectively. The appropriate stability notion for a singular point on for instance the positive first axis is S-stability, because (x_A, x_B, x_C) is restricted to lie in S, the closed first octant in $\mathbb{R}^3$.

In section 3 we specialize to linear gradient fields on $\mathbb{R}^n_\nu$.

In section 4 we turn to the geometric aspects of gradient dynamical systems. Among other things we analyse the properties of Killing gradient fields.

1. CAUSALITY SUBSETS

Let $(M, g = \langle\ ,\ \rangle)$ be a semi-Riemannian manifold of dimension $n \geq 1$ and of differentiability class $C^r, r \geq 2$. Let X be a vector fields of class C^s, $1 \leq s \leq r-1$. To the triple (M, g, X) we associate subsets of M, called *causality subsets*:

$$M_X^+ = \{p \in M | \langle X, X\rangle_p > 0\} \quad M_X^- = \{p \in M | \langle X, X\rangle_p < 0\}$$
$$S_X = \{p \in M | X_p = 0\} \qquad M_X^0 = \{p \in M\backslash S_X | \langle X, X\rangle_p = 0\}.$$

Furthermore, we define a function of differentiability class C^s called the causality function

$$K_X : M \to \mathbf{R}, \qquad p \to \langle X, X\rangle_p.$$

Note that M is the disjoint union of M_X^+, M_X, M_X^0, S_X and that $M_X^+ = K_X^{-1}(\mathbf{R}_+)$, $M_X^- = K_X^{-1}(\mathbf{R}_-)$, i.e. M_X^+ and M_X^- are open subsets of M.

The maximal integral curve $\phi_p : J(p) \to M$ through $p \in M$ is defined on an open interval $J(p) =]t^-(p), t^+(p)[$ containing 0 in $\mathbf{R}$. The global flow $\phi : \mathcal{D}(X) \to M$, where $\mathcal{D}(X) = \{(t,p) \in \mathbb{R} \times M | t^-(p) < t < t^+(p)\}$, is a mapping of class C^s defined on the open subset $\mathcal{D}(X)$ of $\mathbf{R} \times M$,

LEMMA 1.1. *Let X be a vector field of class C^s. If $\mathrm{d}\,K_X(X)_p > 0$ for all $p \in M_X^0$, then the flow has the following property:*

For $q \in M_X^0 \cup M_X^+$: $\quad \phi_q(t) \in M_X^+$ for all $t \in]0, t^+(q)[$.

Proof. If $q \in M_X^0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(K_X \circ \phi_q)(0) = \mathrm{d}K_X(X)_q > 0$$

Consequently there is a $\delta > 0$ such that $\phi_q(t) \in M_X^+$ for $t \in]0,\delta[$. It therefore suffices to prove the lemma for $q \in M_X^+$. Working towards a contradiction suppose that there is a $t_0 \in]0, t^+(q)[$ such that $K_X \circ \phi_q(t_0) = 0$. Now define $z_1 = \inf\{z \in]0, t^+(q)[\mid \dot{K}_X \circ \phi_q(t) = 0\} > 0$. By continuity of $K_X \circ \phi_q$ we have $\phi_q(t_1) \in M_X^0$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}(K_X \circ \phi_q)(t_1) = \mathrm{d}K_X(X)_{\phi_q(t_1)} > 0$$

there is a $t_2 \in]0, t_1[$ such that $K_X \circ \phi_q(t_2) < 0$. This contradicts connectedness of $[0, z_1[$, since $K_X \circ \phi_q(0) > 0$ and the lemma is proved. ∎

Recall, that $p \in M$ is an ω(resp.α)-limit point of $q \in M$ iff $t^+(q) = +\infty (t^-(q) = -\infty)$ and there exists a sequence $(t_n)_{n\in\mathbf{N}}$ in $J(q)$ such that $t_n \to +\infty$ and $\phi_q(t_n) \to p$ for $n \to +\infty$. The set of $\omega(\alpha)$-limit points are denoted by $\omega(q)(\alpha(q))$. Define

$$L_\omega(X) = \bigcup_{q\in M} \omega(q) \qquad \text{and} \qquad L_\alpha(X) = \bigcup_{q\in M} \alpha(q).$$

A nonconstant integral curve $\phi_q : J(q) \to M$ is called recurrent if it is ω-recurrent i.e. $\phi_q(J(q)) \subseteq \omega(q)$ and/or α-recurrent i.e. $\phi_q(J(q)) \subseteq \alpha(q)$. In this case we also say that the orbit $\phi_q(J(q))$ is recurrent. In what follows f is a function on M of differentiability class $c^k, 2 \leq k \leq r$.

THEOREM 1.2. *Suppose that either*

1. $\mathrm{d}K_{\mathrm{grad}\, f}(\mathrm{grad}\, f)_p > 0$ for all $p \in M^0_{\mathrm{grad}\, f}$

or that

2. $\mathrm{d}K_{\mathrm{grad}\, f}(\mathrm{grad}\, f)_p < 0$ for all $p \in M^0_{\mathrm{grad}\, f}$.

Then a) $L_\omega(\mathrm{grad}\, f) = S_{\mathrm{grad}\, f}$ *and b)* $L_\alpha(\mathrm{grad}\, f) = S_{\mathrm{grad}\, f}$.

Proof of 1a. If $L_\omega(\mathrm{grad}\, f) = \emptyset$ then also $S_{\mathrm{grad}\, f} = \emptyset$. Otherwise consider $p \in L_\omega(\mathrm{grad}\, f)$ and assume that $p \notin S_{\mathrm{grad}\, f}$.

CASE I. $p \in L_\omega(\mathrm{grad}\, f) \cap M^+_{\mathrm{grad}\, f}$.

$Q = f^{-1}(f(p)) \cap M^+_{\mathrm{grad}\, f}$ is a hypersurface in M, since $\langle \mathrm{grad}\, f, \mathrm{grad}\, f\rangle q > 0$ and therefore $\mathrm{d}f_q \neq 0$ for all $q \in Q$. This also shows, that $\mathrm{grad}\, f_p \notin T_pQ = \ker \mathrm{d}f_p$. By the Tubular Flow theorem there is a diffeomorphism $\varphi = (\varphi_1, \ldots, \varphi_n) : U \to \varphi(U) = \{\underline{x} \in \mathbf{R}^n \mid |x_i| < \delta > 0$, of class C^{k-1} with

$$p \in U,\ T\varphi(\mathrm{grad}\, f \mid U) \equiv (1, 0, \ldots 0) \quad \text{and} \quad Q \cap U = \{q \in U \mid \varphi_1(q) = 0\},$$

where $T\varphi$ denotes the tangential of $\varphi \cdot \phi$ is the global flow of grad f. Since $p \in L_\omega(\text{grad } f)$ there exists a $q \in M$ and a sequence $(t_m)_{m\in\mathbf{N}}$ in $J(q)$ such that $z_m \to +\infty$ and $\phi_q(t_m) \to p$ for $m \to +\infty$. Using this sequence we can find two positive real numbers $s_1 < s_2$ with $\phi_q(s_1), \phi_q(s_2) \in Q$. But for $t > s_1, f \circ \phi_q$ increases strictly, since

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \phi_q)(t) = \langle \text{grad } f, \text{grad } f \rangle_{\phi_q(t)} > 0$$

by Lemma 1.1. This contradicts $f \circ \phi_q(s_1) = f \circ \phi_q(s_2) = f(p)$.

CASE II. $p \in L_\omega(\text{grad } f) \cap M^0_{\text{grad } f}$.

Note, that $\mathrm{d}\, K_{\text{grad } f_p} \neq 0$ for all $p \in M^0_{\text{grad } f} = K_{\text{grad } f}^{-1}(O) \setminus S_{\text{grad } f}$. Therefore $M^0_{\text{grad } f}$ is a hypersurface in M. Again using the Tubular Flow Theorem there is a diffeomorphism $\varphi : U \to \varphi(U) = \{\underline{x} \in \mathbf{R}^n \mid |x_i| < \delta\}, \delta > 0$, of class C^{k-1} with

$$p \in U, T\varphi(\text{grad } f \mid U) \equiv (1,0,0\ldots 0) \quad \text{and} \quad M^0_{\text{grad } f} \cap U = \{q \in U \mid \varphi_1(q) = 0\}.$$

Since $p \in L_\omega(\text{grad } f)$ there is a $q \in M$ and a sequence $(t_m)_{m\in\mathbf{N}}$ in $J(q)$ such that $t_m \to +\infty$ and $\phi_q(t_m) \to p$ for $m \to +\infty$. Now choose $s_1 \in J(q)$ with $\phi_q(s_1) \in M^0_{\text{grad } f} \cap U$. Since $\phi_q(t_m) \to p$ for $m \to +\infty$ there is an $s_2 > s_1$ such that $\phi_q(s_2) \in U \cap M^0_{\text{grad } f}$. By assumption 1:

$$\frac{\mathrm{d}}{\mathrm{d}t}(K_{\text{grad } f} \circ \phi_q)(s_2) = \mathrm{d}\, K_{\text{grad } f}(\text{grad } f)_{\phi_q(s_2)} > 0.$$

We can therefore find $s_3 \in]s_1, s_2[$ with $\phi_q(s_3) \in M_{\text{grad } f}$ contradicting Lemma 1.1.

CASE III. $p \in L_\omega(\text{grad } f) \cap M_{\text{grad } f}$.

Since $p \in L_\omega(\text{grad } f)$ there exist a $q \in M$ and a sequence $(t_m)_{m\in\mathbf{N}}$ in $J(q)$ with the property, that $t_m \to +\infty$ and $\phi_q(t_m) \to p$ for $m \to +\infty$. If there exists a $t_0 \in J(q)$ such that $\phi_q(t_0) \in M^0_{\text{grad } f} U M^+_{\text{grad } f}$, we have according to Lemma 1.1., that $\phi_q(t) \in M^+_{\text{grad } f}$ for $t > t_0$. But $M_{\text{grad } f} \subseteq M \setminus M^+_{\text{grad } f}$ is an open neighbourhood of p. Therefore $\phi_q(t_m)$ cannot converge to p for $m \to +\infty$. We conclude, that $\phi_q(t) \in M^-_{\text{grad } f}$ for all $t \in J(q)$.

Now define $Q = f^{-1}(f(p)) \cap M_{\text{grad } f}$. Then we reach a contradiction in the same way as in case I, the only change being, that $f \circ \phi_q$ decreases. Consequently we cannot have $p \notin S_{\text{grad } f}$ and a) follows.

In $(N, k) = (M, -g)$ we have $\text{grad}_N f = -\text{grad}_M f, K^N_{\text{grad } f} = -K^M_{\text{grad } f}$. Therefore

$$L_\alpha(\text{grad}_M f) = L_\omega(\text{grad}_N f) \overset{\text{a}}{=} S_{\text{grad}_N f} = S_{\text{grad}_M f} \text{ and b) is proved.}$$

Applying 1 to $-f$ 2 follows. ■

In the Riemannian case, 1 and 2 of Theorem 1.2. are trivially satisfied, since $M^0_{\text{grad } f} = \emptyset$ and then the result is wellknown (see e.g. [5] p. 13).

$\mathcal{D}(X)_t$ denotes the domain of ϕ_t, $t \in \mathbb{R}$. Recall, that $p \in M$ is a nonwandering point of X iff $t^+(p) = +\infty$ and for all neighbourhoods U of p and all $t_0 \geq 0$ there is a $t > t_0$ such that $U \cap \phi_t(\mathcal{D}(X)_t \cap U) \neq \emptyset$ (cf. [1] p. 513). $\Omega(X)$ denotes the set of all nonwandering points of X.

THEOREM 1.3. *Suppose 1 or 2 of Theorem 1.2. holds. Then* $\Omega(\text{grad } f) = S_{\text{grad } f}$.

The proof of this Theorem is very similar to the proof of Theorem 1.2. and is left to the reader. ■

REMARK 1.4. If grad f is complete (e.g. if M is compact), then Theorem 1.2. follows from Theorem 1.3., since then

$$\overline{L_\omega(\text{grad } f)} \subseteq \Omega(\text{grad } f),\ \overline{L_\alpha(\text{grad } f)} = \overline{L_\omega(\text{grad}(-f))} \subseteq \Omega(\text{grad } -f)$$

by [1] p. 514.

EXAMPLE 1.5. $M = \mathbb{R}^2_1$ i.e. $g = -dx_1^2 + dx_2^2, x = (x_1, x_2) \in \mathbb{R}^2$. Define $f : M \to \mathbb{R}$ by $(x_1, x_2) \to x_1^2 + \frac{1}{2}x_2^2$. Then grad $f = (-2x_1, x_2)$ and $M^0_{\text{grad } f} = \{x \in R^2 \backslash \{\underline{0}\} \mid x_2 = \pm 2x_1\}$. Also 1 of Theorem 1.2. holds and

$$\Omega(\text{grad } f) = L_\omega(\text{grad } f) = L_\alpha(\text{grad } f) = S_{\text{grad } f} = \{\underline{0}\}$$

see figure 1.

PROPOSITION 1.6. ϕ *denotes the global flow of* grad f. *Suppose, that* $\phi_q : J(q) \to M$ *is* ω*-recurrent for* grad f, *where* $q \in M$.

If $q \in M^+_{\text{grad } f}(\text{resp.} M^-_{\text{grad } f})$, *there exists a* $t \in]0, t^+(q)[$ *such that* $\phi_q(t) \in M^-_{\text{grad } f}(\text{resp.} M^+_{\text{grad } f})$.

Proof. Let $q \in M^+_{\text{grad } f}$. Assume $\phi_q(t) \in M \backslash M^-_{\text{grad } f}$ for all $t \in]0, t^+(q)[$. We reach a contradiction as follows.

Consider the hypersurface $Q = f^{-1}(f(q)) \cap M^+_{\text{grad } f}$ in M and a Tubular Flow $\varphi : U \to \varphi(U) = \{x \in \mathbb{R}^n \mid |x_i| < \delta\}, \delta > 0$, with

$$q \in U,\ T\varphi(\text{grad } f_{|U}) \equiv (1, 0, \ldots 0), Q \cap U = \{q \in U | \varphi_1(q) = 0\}.$$

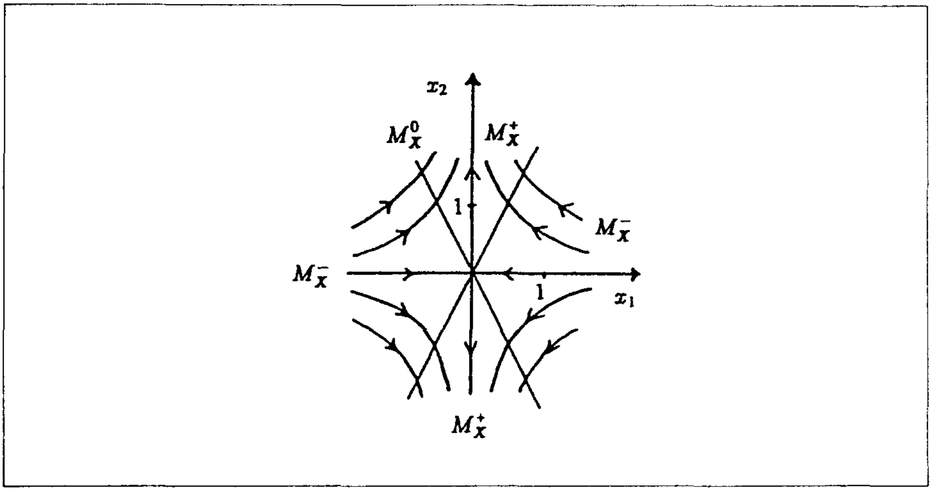

Fig. 1. Orbits and causality subsets for the gradient $X(x_1, x_2) \triangleq \text{grad } f(x_1, x_2) = (-2x_1, x_2)$ **on** $M = \mathbf{R}_1^2$.

Now $q \in \omega(q)$, since $\phi_q^{\cdot}$ is ω-recurrent. As in Theorem 1.2. we can find an $s > 0$ such that $\phi_q(s) \in Q$. Since $q \in M^+_{\text{grad } f}$ and $\phi_q(t) \in M \backslash M_{\text{grad } f}$ for $t > 0$, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \phi_q)(t) = \langle \text{grad } f, \text{grad } f \rangle_{\phi_q(t)} \begin{cases} \geq 0 & \text{for } t > 0 \\ > 0 & \text{for } t = 0 \end{cases}.$$

Therefore $f \circ \phi_q(s) > f(q)$, contradicting the fact that $q, \phi_q(s) \in Q$. Consequently there exists a $t \in]0, t^+(q)[$ such that $\phi_q(t) \in M^-_{\text{grad } f}$. This proves the first statement. The statement in ( ) follows by applying the first statement to $(M, -g)$ and $-f$. ∎

REMARK 1.7. Suppose, that $\phi_q : J(q) \to M$ is ω-recurrent for grad f as in Proposition 1.6. Then we can find an increasing sequence $(t_m)_{m \in \mathbf{N}}$ in $J(q)$, such that

$$\phi_q(t_m) \in \begin{cases} M^+_{\text{grad } f} & \text{for } m \text{ even} \\ M^-_{\text{grad } f} & \text{for } m \text{ odd} \end{cases}$$

according to Proposition 1.6. So the orbit containing q must repeatedly leave and return to $M^+_{\text{grad } f}, M^-_{\text{grad } f}$ and $M^0_{\text{grad } f}$ respectively.

Note also, that every nonconstant ω-recurrent orbit must intersect $M^0_{\text{grad } f}$.

EXAMPLE 1.8. $M = \mathbf{R}_1^2$. Define $f : M \to \mathbf{R}$ by $(x_1, x_2) \to x_1 x_2$. Then grad $f(x_1, x_2) = (-x_2, x_1)$. Conditions 1 and 2 of Theorem 1.2. are both violated. Since (0, 0) is a center we have

$$S_{\text{grad } f} = \{(0,0)\} \neq \Omega(\text{grad } f) = L_\omega(\text{grad } f) = L_\alpha(\text{grad } f) = M.$$

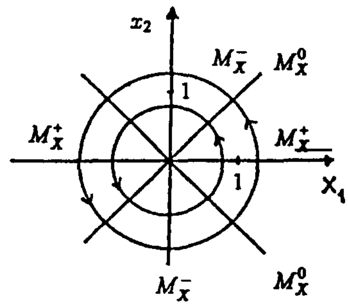

Fig. 2. Orbits and causality subsets for the gradient field $X(x_1,x_2) \triangleq \text{grad } f(x_1,x_2) = (-x_2,x_1)$ **on** $M = \mathbb{R}_1^2$.

This example illustrates Proposition 1.6. and Remark 1.7., see figure 2.

Suppose, there are sequences $q = (q_i)_{i=1}^{k}$ in M and $p = (p_i)_{i=1}^{k+1}$ in S_X, with the property that

$$p_1 = p_{k+1}, \alpha_X(q_i) = p_i \quad \text{and} \quad \omega_X(q_i) = p_{i+1}, i \in \{1,\ldots,k\}, k \in \mathbb{N}.$$

Define $\gamma_i = \phi_{q_i}(J(q_i))$ and $G_X = \left[\bigcup_{i=1}^{k+1} p_i\right] \cup \left[\bigcup_{i=1}^{k} \gamma_i\right]$.

G_X is called an X-loop. In the same spirit as Proposition 1.6. we have

PROPOSITION 1.9. *Suppose, that* grad f *has a* grad f*-loop* $G_{\text{grad } f}$.
If $G_{\text{grad } f} \cap M^+_{\text{grad } f} \neq \emptyset$ (resp. $G_{\text{grad } f} \cap M^-_{\text{grad } f} \neq \emptyset$) *then also* $G_{\text{grad } f} \cap M^-_{\text{grad } f} \neq \emptyset$ (resp. $G_{\text{grad } f} \cap M^+_{\text{grad } f} \neq \emptyset$).

Proof. Given that $G_{\text{grad } f} \cap M^+_{\text{grad } f} \neq 0$, assume $G_{\text{grad } f} \cap M^-_{\text{grad } f} \neq \emptyset$. Now choose $j \in \{1,\ldots,k\}$ and $t_j \in J(q_i)$ such that $\phi_{q_j}(t_j) \in M^+_{\text{grad } f}$. For $t \in J(q_i)$, $i \in \{1,\ldots,k\}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \phi_{q_i}(t) = \langle \text{grad } f, \text{ grad } f\rangle_{\phi_{q_i}(t)} \begin{cases} > 0 & t = t_i, i = j \\ \geq 0 & \text{otherwise} \end{cases}$$

Therefore $f(p_1) \leq \cdots \leq f(p_j) < f(p_{j+1}) \leq \cdots f(p_{k+1}) = f(p_1)$, a contradiction. Consequently we must have $G_{\text{grad } f} \cap M^-_{\text{grad } f} \neq \emptyset$. This proves the first statement. The statement in ( ) follows by applying the first statement to $(M,-g)$. ■

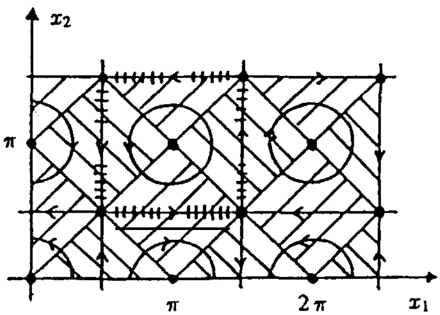

Fig. 3. Orbits and causality subsets for the gradient grad $f(x_1, x_2) = (\quad \cos x_1 \sin x_2,$ $\sin x_1 \cos x_2)$ **on** $M = \mathbf{R}_1^2$.

EXAMPLE 1.10. $M = \mathbf{R}_1^2$. Define $f : M \to \mathbf{R}$ by $(x_1, x_2) \to \sin x_1 \sin x_2$. Then grad $f = (-\cos x_1 \sin x_2, \sin x_1 \cos x_2)$ and there exists a grad f-loop $G_{\text{grad } f}$ such that $G_{\text{grad } f} \cap M^+_{\text{grad } f} \neq \phi$, as illustrated by Figure 3.

$M^+_{\text{grad } f}$: , $M_{\text{grad } f}$: , $G_{\text{grad } f}$:

• : singular point of grad f.

Proposition 1.9. asserts that $G_{\text{grad } f} \cap M_{\text{grad } f} \neq \emptyset$.

The following notation turns out to be convenient

$$M^0_{\text{grad } f} = M \setminus M^+_{\text{grad } f} \qquad M^{0+}_{\text{grad } f} = M \setminus M_{\text{grad } f}.$$

Wellknown ideas lead to

PROPOSITION 1.11. *Let* K *be compact in* $M \cdot \phi$ *denotes the global flow of* grad f. *If* $K \cap M^{0-}_{\text{grad } f}$ *is positively invariant for* ϕ, *then for every* $q \in K \cap M^0_{\text{grad } f}$ *there exists a* $c \in f(M)$ *such that*

$$\emptyset \neq \omega(q) \subseteq f^{-1}(c) \cap K \cap (M^0_{\text{grad } f} \cap S_{\text{grad } f}).$$

Proof. Suppose that $K \cap M^0_{\text{grad } f}$ is positively invariant for ϕ and that $q \in K \cap M^0_{\text{grad } f}$. Note, that $K \cap M^0_{\text{grad } f}$ is compact. We conclude that $\omega(q) \neq \emptyset$ and $\omega(q) \subseteq K \cap M^0_{\text{grad } f}$

It therefore remains to prove that $K_{\text{grad } f} \equiv 0$ and $f \equiv c$ on $\omega(q)$ for some $c \in f(M)$. Now

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \phi_q)(t) = \langle \text{grad } f, \text{grad } f \rangle_{\phi_q(t)} \leq 0$$

for $t > 0$. So $f \circ \phi_q$ decreases for $t > 0$. Since f is bounded on K, $c = \inf\{f \circ \phi_q(t) \mid t > 0\}$ exists and $f|_{\omega(q)} \equiv c$. $\omega(q)$ is invariant under ϕ. For $p \in \omega(q)$ we therefore compute

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \phi_p)(0) = \langle \text{grad } f, \text{grad } f \rangle_p = K_{\text{grad } f}(p)$$

and the proposition is proved. ∎

REMARK 1.12. Let $h : M \to \mathbf{R}_+$ be a function of class C^{k-1}. Then $\omega_{\text{grad } f}(q) = \omega_{h \cdot \text{grad } f}(q)$ for $q \in M$. Using this equality we leave it to the reader to formulate analogues of Theorem 1.2. and Propositions 1.6., 1.9. and 1.11. for a vector field $h \cdot \text{grad } f$.

2. SECTOR STABILITY

S is a subset of M and p a singular point of X.

DEFINITION 2.1. *p is positively S-stable for X if for every neighbourhood U of p there is a neighbourhood V of p such that for all q in $V \cap S$ we have that $t^+(q) = +\infty$ and $\phi_q(t) \in U \cap S$ for all $t > 0$. If in addition V can be chosen so that for all q in $V \cap S$, $\phi_q(t) \to p$ for $t \to +\infty$, then p is called positively asymptotically S-stable for X. p is negatively (asymptotically) S-stable for X iff p is positively (asymptotically) S-stable for $-X$.*

Notice that M-stability = stability in the sense of Lyapounov.

DEFINITION 2.2. A function $L : U \to \mathbf{R}$ of differentiability class C^1 defined on an open neighbourhood U of p and satisfying

i) $L(p) = 0$
ii) $L(q) > 0$, $q \in U \cap S \backslash \{p\}$
iii) $X_q[L] \leq 0$, $q \in U \cap S \backslash \{p\}$

is called a *sector Lyapounov function for S* in p.

If furthermore

$$X_q[L] < 0, \; q \in U \cap S \backslash \{p\}$$

L is a *strict sector Lyapounov function for S* in p.

Notice, that a (strict) sector Lyapounov function for M in p is a (strict) Lyapounov function for p in the usual sense.

We need a slight extension of

LYAPOUNOVS THEOREM. *Suppose S is a closed positively invariant subset of M.*

1. If X has a sector Lyapounov function $L : U \to \mathbb{R}$ for S in p then p is positively S-stable.

2. If $L : U \to \mathbb{R}$ is a strict sector Lyapounov function for S in p, then p is positively asymptotically S-stable.

The proof is similar to that in [3] p. 131. ∎

PROPOSITION 2.3. *Let p be an isolated singular point of* grad f *and W an open neighbourhood of p in M.*

a) If $\mathrm{d}\,K_{\mathrm{grad}\,f}(\mathrm{grad}\,f)_q > 0 \quad (< 0)$ for all $q \in M^0_{\mathrm{grad}\,f} \cap W$, then

1. p is a strict local maximum (minimum) of $f|_{M^{0-}_{\mathrm{grad}\,f}}$ iff
p is positively (negatively) asymptotically $M^{0+}_{\mathrm{grad}\,f}$-stable

2. p is a strict local maximum (minimum) of $f|_{M^{0}_{\mathrm{grad}\,f}}$ iff
p is negatively (positively) asymptotically $M^{0}_{\mathrm{grad}\,f}$-stable

Proof of 1. Let V be a compact neighbourhood of p such that $V \subset W$ and p is the only singular point of grad f in V.

« $\Rightarrow$ » : Since p is a strict local maximum of $f|_{M^{0-}_{\mathrm{grad}\,f}}$ there is an open neighbourhood $U \subseteq W$ of p such that $f(q) < f(p)$ for all $q \in U \cap M^{0+}_{\mathrm{grad}\,f} \backslash \{p\}$. Define $L(q) = f(p) - f(q), q \in U$. Then L is a sector Lyapounov function for $M^{0+}_{\mathrm{grad}\,f}$ in p. By Lyapounovs Theorem p is positively $W^{0+}_{\mathrm{grad}\,f\,|_W}$-stable, since $W^{0+}_{\mathrm{grad}\,f|_W}$ is positively invariant for grad $f|_W$ by Lemma 1.1. But then p is also $M^{0+}_{\mathrm{grad}\,f}$-stable. We conclude, that there exists a neighbourhood V_1 of p such that

$$(*) \qquad \text{For all } q \text{ in } V_1 \cap M^{0+}_{\mathrm{grad}\,f}, \phi_q(t) \text{ is defined and in } V \cap M^{0+}_{\mathrm{grad}\,f} \text{ for all } t > 0, \text{ where } \phi \text{ denotes the global flow of grad } f.$$

Suppose for contradiction, that p is not positively asymptotically $M^{0+}_{\mathrm{grad}\,f}$-stable. Then there exists a $q \in V_1 \cap M^{0+}_{\mathrm{grad}\,f}$, and an open neighbourhood $V_2 \subset V$ of p and a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that $\phi_q(t_n) \in V \backslash V_2$ for all $n \in \mathbb{N}$. Take a convergent subsequence $(\phi_q(t_{n_k}))$ with limit point $q_0 \in V \cap (M^0_{\mathrm{grad}\,f} \cap M^+_{\mathrm{grad}\,f})$. From $(*)$ and Lemma 1.1. we deduce, that

$$(**) \qquad f \circ \phi_q(t) < f(q_0) \qquad \text{for all } t > 0.$$

By Lemma 1.1. there exists a $b > 0$ such that $\phi_{q_0}(t) \in M^+_{\text{grad } f}$ for all $t \in]0, b[$. Now choose $t_1 \in]0, b[$. Then $f \circ \phi_{q_0}(t_1) > f(q_0)$. By continuity of $f \circ \phi$, there is an open neighbourhood $I_1 \times U_1$ of (t_1, q_0) in $\mathcal{D}(\text{grad } f)$ such that $f \circ \phi(t, q_1) > f(q_0)$ for all (t, q_1) in $I_1 \times U_1$. Taking $\phi_q(t_{n_j})$ in U_1 we have

$$f \circ \phi(t_1, \phi(t_{n_j}, q)) = f \circ \phi(t_1 + t_{n_j}, q) > f(q_0)$$

contradicting $(**)$.

« $\Leftarrow$ » : Let p be positively asymptotically $M^{0+}_{\text{grad } f}$-stable and V_1 a neighbourhood of p such that for all q in $V_1 \cap M^{0+}_{\text{grad } f}$, $t^+(q) = +\infty, \phi_q(t) \in V \cap M^{0+}_{\text{grad } f}$ for $t > 0$ and $\phi_q(t) \to p$ for $t \to +\infty$. If p is not a strict local maximum of $f|_{M^{0\cdot}_{\text{grad } f}}$ there exists a $q_1 \in V_1 \cap M^{0+}_{\text{grad } f} \backslash \{p\}$ such that $f(q_1) \geq f(p)$. Now $f \circ \phi_{q_1}$ increases strictly for $t > 0$, since $\phi_{q_1}(t) \in W^+_{\text{grad } f}|_W$ for $t > 0$ by Lemma 1.1. Consequently

$$f(p) = \lim_{t \to +\infty} f \circ \phi_{q_1}(t) > f \circ \phi_{q_1}(0) = f(q_1).$$

a contradiction. Thus p is a strict local maximum of $f|_{M^{0\cdot}_{\text{grad } f}}$.

This proves the first statement. The statement in ( ) follows by applying the first statement to $-f$. Considering $(M, -g)$ we can prove 2 using 1. ■

If M is Riemannian, then a) in Proposition 2.3. is trivially satisfied and then Proposition 2.3. asserts, that we have the wellknown property (see e.g. [2] p. 200):

$(*)$
p is a strict local maximum (minimum) of f iff
p is positively (negatively) asymptotically stable.

EXAMPLE 2.4. $M = \mathbb{R}^2_1$. Define $f : M \to \mathbb{R}$ by $(x_1, x_2) \to 2x_1^2 - x_2^2$. We compute grad $f = (-4x_1, -2x_2)$. Then a) in Proposition 2.3. holds with $W = M$ and $(0,0)$ is a strict maximum of $f|_{M^{0\cdot}_{\text{grad } f}}$. By Proposition 2.3.1. $(0,0)$ is positively asymptotically $M^{0+}_{\text{grad } f}$-stable, see figure 4.

Similar to an instability Theorem of Lyapounov (see e.g. [2] p. 199) we have

PROPOSITION 2.5. *Let* p *be an isolated singular point of* grad f *and* W *an open neighbourhood of* p.

a) If $\mathrm{d}\, K_{\text{grad } f}(\text{grad } f)_q > 0$ (resp. < 0) *for all* q *in* $M^0_{\text{grad } f} \cap W$ *and*

1. every neighbourhood V *of* p *contains a* q *in* $V \cap M^+_{\text{grad } f}$ *such that* $f(q) > f(p)(f(q) < f(p))$, *then* p *is positively (resp. negatively) unstable for* grad f *in the sense of Lyapounov.*

2. every neighbourhood V *of* p *contains a* q *in* $V \cap M^-_{\text{grad } f}$ *such that* $f(q) > f(p)(f(q) < f(p))$, *then* p *is negatively (resp. positively) unstable for* grad f *in the sense of Lyapounov.*

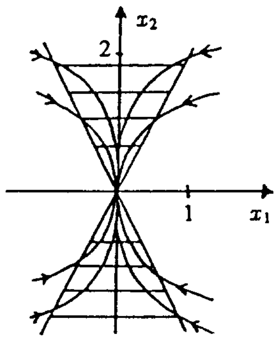

Fig. 4. Orbits for the gradient grad $f(x_1, x_2) = (-4x_1, -2x_2)$ **on** $M = \mathbb{R}^2_1$. $M^{0+}_{\text{grad } f}$ **is dashed.**

Proof of 1. Consider a chart (V, ψ) with $p \in V \subseteq W$ such that $\psi(p) = 0, \overline{B_1(0)} \subseteq \psi(V)$ and p is the only singular point of grad f in V. Here $B_1(0)$ is the open unit ball of $\mathbb{R}^n$ and B the counterimage by ψ of its closure. For every $\delta \in]0, 1[$ there exists a q in $\psi^{-1}(B_\delta(0)) \cap M^+_{\text{grad } f}$ such that $f(q) > f(p)$. Letting ϕ denote the global flow of grad f, assume that $\phi_q(t) \in B \cap M^+_{\text{grad } f}$ for all t in $]0, t^+(q)[$. We reach a contradiction as follows.

We must have $t^+(q) = +\infty$. Choose $\rho \in]0, \delta[$ such that $f(q_1) < f(q)$ for all q_1 in $\psi^{-1}(B_\rho(0))$. Using the assumption we see that $f \circ \phi_q(t) > f \circ \phi_q(0) = f(q)$ for $t > 0$. We therefore have:

$$(*) \qquad \phi_q(t) \in \psi^{-1}(\overline{B_1(0)} \setminus B_\rho(0)) \cap M^+_{\text{grad } f} \quad \text{for } t > 0.$$

Let $(t_m)_{m \in \mathbb{N}}$ be a sequence in $\mathbb{R}_+$ with $t_m \to +\infty$ for $m \to +\infty$. $(\phi_q(t_m))_{m \in \mathbb{N}}$ has a convergent subsequence $(\phi_q(t_{m_i}))$ with limit point $q_0 \in \psi^{-1}(\overline{B_1(0)} \setminus B_\rho(0)) \cap (M^0_{\text{grad } f} \cup M^+_{\text{grad } f})$. According to $(*)$ we have

$$(**) \qquad f \circ \phi_q(t) < f(q_0) \qquad \text{for} \quad f > 0.$$

By Lemma 1.1. there exists a $b > 0$ such that $\phi_{q_0}(t) \in M^+_{\text{grad } f}$ for all $t \in]0, b[$. Thus we can find a $t_1 \in]0, b[$ such that $f \circ \phi(t_1, q_0)$ and hence an open neighbourhood $I_1 \times U_1$ of (t_1, q_0) in $\mathcal{D}(\text{grad } f)$ with $f \circ \phi(t, q_1) > f(q_0)$ for $(t, q_1) \in I_1 \times U_1$. Choose t_{m_j} such that $\phi(t_{m_j}, q) \in U_1$. Then

$$f \circ \phi(t_1, \phi(t_{m_j}, q)) = f \circ \phi(t_1 + t_{m_j}, q) > f(q_0)$$

contradicting $(**)$. We conclude that there exists a $\tilde{t} \in]0, t^+(q)[$ such that $\phi_q(\tilde{t}) \notin B \cap M^+_{\text{grad } f}$.

Now define $A = \{t \in]0, t^+(q)[\mid \phi_q(t) \notin M^+_{\text{grad } f} \cap B\}$ and $s = \inf A > 0$. Then $\phi_q(s) \in B$. Since $M^-_{\text{grad } f}$ is open we cannot have $\phi_q(s) \in M^-_{\text{grad } f}$. If $\phi_q(s) \in M^0_{\text{grad } f} \cap B$, then according to a) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(K_{\text{grad } f} \circ \phi_q\right)(s) > 0$$

contradicting the definition of s. Consequently $\phi_q(s) \in M^+_{\text{grad } f}$. Since $M^+_{\text{grad } f}$ is open, we can find a $t \in]s, t^+(q)[$ such that $\phi_q(t) \notin B$, thereby proving the first statement.

The statement in ( ) follows by applying the first statement to $-f$. Considering $(M, -g)$ we can prove 2 using 1. ∎

EXAMPLE 1.5. illustrates Propositions 2.5. Example 1.8. shows, that condition a) of Proposition 2.5. cannot be omitted.

Suppose now, that M and X are smooth and that X has a singular point p. Let (V, φ) be a chart around p with coordinates $(y_1, \ldots, y_n) \in \varphi(V) \subseteq \mathbf{R}^n$. Then we compute using Einsteins summation convention

$$(*) \qquad \mathrm{d}K_{X_p} = 0 \text{ and } \frac{\partial^2 K_{X \circ \varphi^{-1}}}{\partial y_k \partial y_\ell}(\varphi(p)) = 2 g_{ij_p} \cdot \frac{\partial (X^\varphi)^i}{\partial y_k} \cdot \frac{\partial (X^\varphi)^j}{\partial y_\ell} \varphi(p)$$

where X^φ is the local representative of X and $k, \ell \in \{1, \ldots, n\}$. Thus we see

p is a nondegenerate critical point of K_X iff

p is elementary $\left(i.e. \quad \det\left\{\frac{\partial (X^\varphi)^i}{\partial y_k}(\varphi(p))\right\} \neq \right)$.

If p is an elementary singular point of X we see from $(*)$, that the index of K_X in p equals the index ν of g. In this case there exists by the Morse Lemma an open neighbourhood U of p in M and a diffeomorphism $\psi : U \to \psi(U) \subseteq \mathbf{R}^n$ such that

$$K_X \circ \psi^{-1}(x_1, \ldots, x_n) = \begin{cases} x_1^2 + \ldots + x_n^2 & \nu = 0 \\ -x_1^2 - \ldots - x_\nu^2 + x_{\nu+1}^2 + \ldots + x_n^2 & 0 < \nu < n \\ -x_1^2 - \ldots - x_n^2 & \nu = n \end{cases}$$

for $(x_1, \ldots, x_n) \in \psi(U)$. When $0 < \nu < n$, $M_X^0 \cap \{p\}$ has the structure of a cone in the vicinity of p.

3. LINEAR GRADIENT FIELDS ON $\mathbf{R}^n_\nu$.

We denote the metric on $\mathbf{R}^n_\nu$ by $\langle\ ,\ \rangle$ and the canonical basis in $\mathbb{R}^n$ by $e_1, \dots, e_n$, $n \in \mathbb{N}$. We will identity $\mathbf{R}^n$ and $T_x\mathbb{R}^n$ via the isomorphism

$$\mathbf{R}^n \to T_x\mathbf{R}^n, \qquad e_i \to \partial_{i_x}$$

where ∂_i is the i-th basis vector field in the chart $(\mathbf{R}^n, 1_{\mathbf{R}^n})$ on $\mathbf{R}^n$ and $x = (x^1, \dots, x^n) \in \mathbf{R}^n$.

S^n_ν is the subset of $L(\mathbf{R}^n, \mathbb{R}^n)$ consisting of selfadjoint linear operators on $\mathbf{R}^n_\nu$. We denote the matrix representation of $S \in L(\mathbf{R}^n, \mathbf{R}^n)$ in the canonical basis by $\{S^j_i\}$ and then

$$(3.1) \qquad S \in S^n_\nu \quad \text{iff} \quad S^j_i \epsilon_j = \langle S(e_i), e_j\rangle = \langle e_i, S(e_j)\rangle = S^i_j \epsilon_i$$

where $i, j \in \{1, \dots, n\}$ and $\epsilon_i = \langle e_i, e_i\rangle$.

PROPOSITION 3.1. *Let $S \in L(\mathbf{R}^n, \mathbf{R}^n)$.*

1. *S is the gradient of a smooth function iff $S \in S^n_\nu$.*
2. *If $S \in S^n_\nu$ then $\exp(tS) \in S^n_\nu$ for all $t \in \mathbf{R}$.*

Proof of 1. We use Einsteins summation convention. The one-form metrically equivalent to $S^\ell_i x^i \partial_\ell$ is $\mu_S = \epsilon_k S^k_i x^i \mathrm{d}x^k$.

« $\Leftarrow$ » : For $S \in S^n_\nu$ we have

$$(*) \qquad \mathrm{d}\mu_S = \epsilon_k S^k_i \mathrm{d}x^i \wedge \mathrm{d}x^k = \Sigma_{1 \le k < i \le n} \left[\epsilon_k S^k_i - \epsilon_i S^i_k\right] \mathrm{d}x^i \wedge \mathrm{d}x^k = 0.$$

By Poincarè's lemma μ_S is exact, so S is a linear gradient field.

« $\Rightarrow$ » : If $\mu_S = \mathrm{d}f$ for some smooth function $f : \mathbf{R}^n \to \mathbf{R}$, then $\mathrm{d}\mu_S = \mathrm{dd}f = 0$. According to $(*)$ we have, that $\epsilon_k S^k_i = \epsilon_i S^i_k$, $i, k \in \{1, \dots, n\}$. By (3.1) $S \in S^n_\nu$.

2. Define a linear map $H^i_j : L(\mathbf{R}^n, \mathbf{R}^n) \to \mathbf{R}, L \to e^{*i}(L(e_j))$ where $e^{*1}, \dots, e^{*n}$ is the dual basis to $e_1, \dots, e_n$ and $i, j \in \{1, \dots, n\}$. Then $H : L(\mathbf{R}^n, \mathbf{R}^n) \to \mathbf{R}$, $L \to \sum_{i,j}(H^j_i(L)\epsilon_j - H^i_j(L)\epsilon_i)^2$ is continuous, so $S^n_\nu = H^{-1}(0)$ is closed. Clearly

$$E_k(t) \triangleq \sum_{k=0}^{m} \frac{t^k}{k!} S^k \in S^n_\nu.$$

Finally $\exp(tS) = \lim_{k\to\infty} E_k(t) \in S^n_\nu$, since S^n_ν is closed. ∎

EXAMPLE 3.2. Suppose, that S is selfadjoint in $\mathbf{R}^2_\nu$, i.e. S is a gradient field by Proposition 3.1. Then all types of elementary singular points can occur when $\nu = 1$; in contrast to the case $\nu = 0$, where improper node, focus and center are impossible.

4. GEOMETRIC ASPECTS OF GRADIENT DYNAMICS

In this section (M, g) is a smooth n-dimensional semi-Rimannian manifold with Levi-Civita connection D. X is a smooth Killing vector field on M. An integral curve for X through a critical point q of K_X is a geodesic (see [4] p. 259), since grad $K_X = -2\mathrm{D}_X X$.

For $q \in M$ and $t \in J(q)$ we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}(K_X \circ \phi_q)(t) = (\langle \mathrm{D}_X X, X\rangle + \langle X, \mathrm{D}_X X\rangle)_{\phi_q(t)} = 0.$$

This shows, that M_X^0, M_X^- and M_X^+ are invariant under the flow ϕ for X. If in addition X is the gradient of a smooth function f on M, Proposition 1.6. asserts: If X has recurrent orbits, then they must be contained in M_X^0.

Let $c \in f(M)$ and suppose $\gamma : I \to M$ is a periodic geodesic through $\gamma(t_0) \in f^{-1}(c), t_0 \in I$. By the conservation lemma

$$\frac{\mathrm{d}\, f \circ \gamma}{\mathrm{d}t}(t) = \langle \text{grad } f, \gamma' \rangle(t) \equiv k \in \mathbf{R} \quad \text{for alle} \quad t \in I.$$

Because γ is periodic, we must have $k = 0$. We conclude: If $\gamma : I \to M$ is a periodic geodesic in M, then $\gamma(I) \subseteq f^{-1}(c)$ for some $c \in f(M)$.

Suppose $\mathrm{d} \in f(M)$ satisfies $K_{\text{grad } f}(p) > 0$ (< 0) for all p in $f^{-1}(\mathrm{d})$; then $f^{-1}(d)$ is a smooth semi-Riemannian hypersurface of M, see e.g. [4] p. 106. Actually this statement can be sharpened in the following way

$$\begin{gathered} f^{-1}(\mathrm{d}) \text{ is a smooth semi-Riemannian hypersurface of } M \\ \text{iff } f^{-1}(\mathrm{d}) \subseteq M^+_{\text{grad } f} \text{ or } f^{-1}(\mathrm{d}) \subset M^-_{\text{grad } f}. \end{gathered}$$

If $\gamma : I \to M$ is a geodesic, tangent to $f^{-1}(\mathrm{d})$ in $\gamma(t_0), t_0 \in I$, then (4.1) shows, that $\gamma(I) \subseteq f^{-1}(\mathrm{d})$. So $f^{-1}(\mathrm{d})$ is a smooth geodesic semi-Riemannian hypersurface of M.

PROPOSITION 4.1. *Let h be a smooth function on M and p a hyperbolic singular point for* grad h. *Then the stable manifold of* grad h *in p, $W^s(p)$, and the unstable manifold of* grad h *in $p, W^u(p)$, are orthogonal in p.*

Proof. Let (V, ψ) be a chart around p with coordinates $(x^1, \ldots, x^n) \in \psi(V) \subseteq \mathbf{R}^n$. We use Einstein's summation convention. Then the local representative $Y^\psi : \psi(V) \to \mathbb{R}^n$ of $Y = \text{grad } h$ is

$$g^{ij} \frac{\partial h}{\partial x_i} e_j.$$

Put $h_{ki} = \frac{\partial^2 h}{\partial x^k \partial x^i}$ $\quad 1 \leq i, k \leq n$.

In the canonical basis in $\mathbf{R}^n$ the matrix representation of $S = \mathrm{D}Y^{\psi}_{\psi(p)}$ is

$$\{g^{ij} h_{ki}\} \qquad 1 \leq j \leq n,\ 1 \leq k \leq n$$

S is selfadjoint with respect to the induced scalar product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$, since

$$\begin{aligned}\langle e_m, S(e_k)\rangle &= \langle e_m, g^{ij} h_{ki} e_j\rangle = h_{km} \\ =\langle S(e_m), e_k\rangle &= \langle g^{ij} h_{mi} e_j, e_k\rangle = h_{mk} \quad m, k \in \{1, \ldots, n\}.\end{aligned}$$

According to [3] p. 108 there is a direct sum decomposition $\mathbf{R}^n = E^s \otimes E^u$ into S-invariant subspaces, such that the eigenvalues of $S|_{E^s}$ have negative real part and the eigenvalues of $S|_{E^u}$ have positive real part.

Furthermore: $T\psi(T_p W^s(p)) = E^s$ and $T\psi(T_p W^u(p)) = E^u$ where again $T\psi$ is the tangential of ψ, see [3] p. 98 & 152.

By [4] p. 261, exercise 18, E^s and E^u are orthogonal in the scalar product induced on $\mathbf{R}^n$; since a cyclic subspace in the unstable summand and a cyclic subspace in the stable summand of the complexified linear operator S are orthogonal in the scalar product induced on $\mathbb{C}^n$. ∎

EXAMPLE 4.2. To illustrate the ideas of the proceding sections consider a smooth function f on $\mathbf{R}^n_\nu$. Then

$$K_{\mathrm{grad}\, f} = \epsilon_j \left(\frac{\partial f}{\partial x_j}\right)^2$$

$$\begin{aligned}\mathrm{d}\, K_{\mathrm{grad}\, f}(\mathrm{grad}\ f) &= 2\,\epsilon_j \frac{\partial f}{\partial x_j} \frac{\partial^2 f}{\partial x_j \partial x_k} \mathrm{d}\, x_k \left(g^{pq} \frac{\partial f}{\partial x_p} \frac{\partial}{\partial x_q}\right) \\ &= 2\,\epsilon_j \epsilon_k \frac{\partial f}{\partial x_j} \frac{\partial f}{\partial x_k} \frac{\partial^2 f}{\partial x_j \partial x_k}\end{aligned}$$

Now define a symmetric tensor s on $\mathbf{R}^n_\nu$ with components

$$s_{jk} = \epsilon_j \epsilon_k \frac{\partial^2 f}{\partial x_j \partial x_k}$$

in the chart $(\mathbf{R}^n, 1_{\mathbf{R}^n})$ on $\mathbb{R}^n$. s is positive definite on an open subset D of $\mathbf{R}^n$, Hence

$$\mathrm{d}\, K_{\mathrm{grad}\, f}(\mathrm{grad}\ f) > 0$$

on $M^0_{\text{grad } f} \cap \mathrm{D}$. In view of Theorem 1.2. this implies that grad f can have no non-constant recurrent orbits contained in D. As a simple application consider the gradient of

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3^2$$

on $\mathbf{R}^3_1$, which has metric tensor $-\mathrm{d}\,x_1^2 + \mathrm{d}\,x_2^2 + \mathrm{d}\,x_3^2$. grad f can have no nonconstant recurrent orbits, e.g. closed orbits, contained in $x_2 > \frac{4}{3}x_3^2 - 1$; as the reader can easily verify by showing that s is positive definite at the singular point at the origin and nondegenerate in the connected open set $x_2 > \frac{4}{3}x_3^2 - 1$. Hence s is positive definite throughout this set.

Consider now the one parameter family of semi-Riemannian gradients grad f^μ on $\mathbf{R}^2_1$, where

$$f^\mu(x_1, x_2) = x_1^4 + \mu^2 x_2^6$$

and $\mu \in \mathbf{R}\backslash\{0\}$. grad f^μ has a degenerate singular point at the origin for every $\mu \in \mathbf{R}\backslash\{0\}$. Compute

$$K_{\text{grad } f^\mu}(x) = -16 x_1^6 + 36 \mu^4 x_2^{10}$$

$$\mathrm{d}\, K_{\text{grad } f^\mu}(\text{grad } f^\mu)_x = 24(16 x_1^8 + 90 \mu^6 x_2^{14})$$

The latter is > 0 on $M\backslash S_{\text{grad } f^\mu}$, in particular on $M^0_{\text{grad } f^\mu}$. Note that $(0, \epsilon) \in M^+_{\text{grad } f^\mu}$ for $\epsilon > 0$. Furthermore $f^\mu(0, \epsilon) > 0$. According to Proposition 2.5.1. the origin is positively unstable for grad f^μ in the sense of Lyapounov. $(\epsilon, 0) \in M^-_{\text{grad } f^\mu}$ and $f^\mu(\epsilon, 0) > 0$ for all $\epsilon > 0$. By Proposition 2.5.2. the origin is negatively unstable for grad f^μ in the sense of Lyapounov.

EXAMPLE 4.3. Recall, that in the theory of relativity an observer field U on a Lorentz manifold (M, g) is a futurepointing timelike unit vector field on $M \cdot U$ is *synchronizable* if there are smooth functions $h : M \to \mathbf{R}_+$ and $t : M \to \mathbf{R}$ such that $U = -h\,\text{grad}\, t$. Note,that $M^0_U = \emptyset$. By remark 1.12.:

$$L_\omega(U) = L_\alpha(U) = \emptyset.$$

In particular U has no recurrent orbits.

REMARK 4.4. An electrical circuit with 1 inductor and 3 capacitors has the metric tensor of a spacetime, see [6]. Thus it can serve as a testground for gradient dynamical systems on spacetimes.

Let $i(M)$ denote the set of all Killing vector fields on M and $I(M)$ the set of all Killing gradient fields on M. By [4] p. 253 $i(M)$ is a finite dimensional Lie algebra over the reals, of dimension at most $\frac{1}{2}n(n+1)$ if M is connected. Clearly $I(M)$ is a subspace of $i(M)$.

PROPOSITION 4.5. *Let M be connected and h a smooth function on M.*

1. *If $X = \text{grad } h$ is a Killing gradient field on M and there exists a $p \in M$ such that $X_p = 0$, then $X \equiv 0$.*
2. $\dim I(M) \leq n$.

Proof of 1. Suppose $X_p = 0$. In a chart (V, ψ) around p with coordinates $(y^1, \dots, y^n) \in \psi(V)$ we have

$$X_{|_V} = g^{ij} \frac{\partial h}{\partial y^j} \partial_i = X^i \partial_i$$

$\mathrm{D}X$ is skewadjoint relative to g (see [4] p. 251). Thus, evaluating in p, we get:

$$\langle \mathrm{D}_{\partial_k} X^i \partial_i, \partial_\ell \rangle = \left\langle \frac{\partial X^i}{\partial y^k} \partial_i, \partial_\ell \right\rangle = \langle g^{ij} h_{kj} \partial_i, \partial_\ell \rangle = h_{k\ell}$$
$$= \langle \mathrm{D}_{\partial_\ell} X^i \partial_i, \partial_k \rangle = -h_{\ell k}, \; k, \ell \in \{1, \dots, n\}.$$

Therefore $\mathrm{D}X_p = 0$. Using [4] p. 253 we get $X \equiv 0$.

2. The linear map $I(M) \to T_pM, X \to X_p$, is injective according to 1. ∎

The following example shows, that equality in Proposition 4.5.2. can in fact occur.

EXAMPLE 4.6. In $\mathbf{R}^n_\nu$ the Killing vector fields are of the form $X : \mathbf{R}^n \to \mathbf{R}^n \; x \to v + S(x)$, where $v \in \mathbf{R}^n$ and S is a skewadjoint linear map, (see e.g. [4] p. 253). The differential of X in 0 is S. If X is a gradient, the proof of Proposition 4.1. asserts that S is selfadjoint and therefore $S = 0$. Clearly, the constant vectorfield $x \to v$ is a gradient. Thus we have according with Proposition 4.5.1. Furthermore $\dim I(M) = n$.

5. CONCLUSION

This paper poses among others the following problem:

For a polynomial function f on $\mathbf{R}^n_\nu$, determine conditions on the coefficients of f in order that the assumption in Theorem 1.2. is fullfilled for grad f or its restriction to some open set. Use for instance the ideas laid down in Example 4.2.

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REFERENCES

[1] ABRAHAM and MARSDEN: *Foundations of Mechanics*, 2. edition, Benjamin-Cummings 1978.
[2] HIRSH and SMALE: *Differential Equations, Dynamical Systems and Linear Algebra*, Academic Press 1974.
[3] M.C. IRWIN: *Smooth Dynamical Systems*, Academic Press 1980.
[4] B. O'NEILL: *Semi-Riemannian Geometry with Applications to Relativitity*, Academic Press 1983.
[5] J. PALIS JR. and W. DE MELO: *Geometric Theory of Dynamical Systems*, Berlin-Heidelberg – New York: Springer Verlag 1980.
[6] S. SMALE: *On The Mathematical Foundations of Electrical Circuit Theory*, Journal of Differential Geometry 7, 1972, p. 193-210.

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